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Strongly Propagative Systems, Generalized Nonselfadjoint Wave Equations and Their Steady-State Solutions

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The first-order systems describing wave propagation in classical physics have many special properties. They may often be reduced to families of wave equations. We discuss certain vector wave equations with matrix potentials which are quasi long range and their relation to the equations of classical physics and the steady-state solutions of these problems. © 1987 Academic Press, Inc.

0. INTRODUCTION

One of the important problems in classical physics is the prediction of the eschatological behavior of a wave transmitting medium or system in which an oscillatory source acts. The wave transmission problems in classical physics and certain quantum physics systems may be cast in Friedrichs–Wilcox form:

$$-iE(x)\,\partial_t u = A(D)u + Bu, \tag{0.1}$$

where $A(D)$ is a first-order symmetric matrix differential operator and B and E are matrices. Examples include Maxwell's equations, the equations of magnetohydrodynamics, the equations of elasticity, crystal optics, and many other phenomena.

In practice, the form (0.1) is considered somewhat unwieldy and can be reduced in many instances to variations of the form

$$\partial_{tt}\mathbf{v} = c\,\Delta\mathbf{v} \tag{0.2}$$

which is simply the classical wave equation. Often, further examination leads to the proposal of various perturbed versions of (0.2) such as

$$\partial_{tt}u = d\,\Delta u + q_1\,\partial_t u + q_2 u. \tag{0.3}$$

Here Δ is the Laplacian.

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We shall seek the answers to two questions in this setting. First, what is the relation between the "simplified" models (0.3) and the systems (0.1) and second, can we then solve the steady-state problem of the opening paragraph for the generalized form of (0.1) related to Eq. (0.3). We shall allow anisotropic forms of (0.3) in our discussion with the condition that the unperturbed medium involved is "strongly propagative" [5].

In order to construct a solution to the steady-state problem mentioned above it will be necessary to introduce several types of function spaces and review certain facts from Schulenberger and Wilcox [5] and Gilliam and Schulenberger [3]. We gather the material from [3, and 5] in Section 1, as well as certain other restrictions, notations, and facts. We shall construct our solutions by means of a *limiting absorption principle*. This method has been used by us for the classical form of (0.1) in [6]. It is perhaps somewhat remarkable since the operators involved are not selfadjoint. Because of the goal of relating the models (0.3) to (0.1), a different proof of the limiting absorption principle is required (the extended version of (0.1) must contain pseudodifferential operator perturbations instead of matrix perturbations).

In Section 2 we develop facts concerning certain subordinate operators related to generalized (anisotropic) versions of (0.2). The main proofs are relagated to two appendices.

In Section 3 the steady-state solutions of certain families of equations of the sort (0.3) are constructed. It is worth remarking that our hypotheses concerning the behavior of the coefficients (at ∞) are somewhat weaker than those studied heretofore in (0.3). We show the existence of solutions to Eq. (0.1).

In Section 4 we consider in an elementary but justifiable fashion, the steady-state problem for the suitable generalized version of (0.1). The fact that $A(D)$ in (0.1) is usually not an elliptic operator plays a limiting role in what can be achieved. See also [6].

Before moving into the main body of the paper we shall review, in a non-rigorous fashion, the relation of the limiting absorption principle to the steady-state existence problem.

Suppose, for example, we wish to construct solutions of steady-state type for a system

$$-i\partial_t f = \Lambda f, \qquad f = f(x, t); \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

To generate steady-state solutions, we induce forced oscillations into the system by adding a term of the form $g(x, t) = e^{-i\lambda t} g(x)$, where $\lambda = \lambda_0 \pm i\varepsilon$:

$$-i\partial_t f = \Lambda f + g(x, t).$$

Of course if g is to be a genuine oscillator, we must take $\varepsilon = 0$. There will not be a unique solution in this case. However, let us assume that a steady-state solution exists, with a form like that of g:

$$f(x, t) = e^{-i\lambda t} f_1(x, \lambda).$$

Inserting $f(x, t)$ into the forced equation gives

$$-i\partial_t e^{-i\lambda t} f_1(x, \lambda) = A e^{-i\lambda t} f_1(x, \lambda) + e^{-i\lambda t} g(x)$$

or

$$-\lambda e^{-i\lambda t} f_1 = e^{-\lambda t i} A f_1 + e^{-\lambda t i} g$$

thus

$$-\lambda f_1 = A f_1 + g$$

or

$$(A - \lambda) f_1 = (-g).$$

If we consider the equation above thinking of f and g as elements of suitable vector spaces then a solution f exists when the operator inverse $(A-\lambda)^{-1}$ can be defined on the vector space containing g. Of course $(A-\lambda)^{-1}$ is simply the resolvent of A and it exists, by definition, when λ is not in the spectrum of A. The difficulty lies in the fact that the problems of interest here contain the real axis in their spectra. We are thus led to consider in what sense the limits

$$\lim_{\varepsilon \to 0} (A - \lambda)^{-1}(-g)$$

may exist. There are really two limits here since $\lambda = \lambda_0 \pm i\varepsilon$. (See the discussion in [2].) It is conceivable that if the topology of the domain of $(A-\lambda)^{-1}$ is *strengthened* and that of the range *weakened* then the set of λ for which $(A-\lambda)^{-1}$ exists may be enlarged to include the real axis or perhaps some portion of it. (If λ_0 is in the point spectrum of A this technique will not work). This idea does indeed hold true. The idea of using the limit(s) above to define a steady-state solution is called the limiting absorption principle. The term "absorption" comes from the fact for $\varepsilon > 0$, $\lambda = \lambda_0 + i\varepsilon$, $g(x, t)$ damps out as $t \to \infty$, x fixed.

Thus the limiting absorption principle is a way of attacking the existence problem for steady-state solutions. Again, the fact that it can be applied in our problem is somewhat remarkable since the appropriate A is almost never selfadjoint (though this is by no means the first time it has been applied to nonselfadjoint problems).

1. Preliminaries

Here we state some assumptions and review some results of [3]. Without loss of practical generality, we suppose $A(D)$ is of the form (see [3])

$$\begin{bmatrix} 0 & B(D)^* \\ B(D) & 0 \end{bmatrix}_{m \times m}, \tag{1.1}$$

where $B(D)$ is a constant coefficient first order $r \times k$ formal differential operator matrix with $B(D)^*$ its formal adjoint, $r+k=m$. We shall define $A(D)$ by means of the Fourier transform Φ (here and below, $n > 2$):

$$(\Phi f)(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(ix \circ p)\, f(x)\, dx = \hat{f}(p). \tag{1.2}$$

Φ is an isomorphism on $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^m)$, the space of smooth rapidly decreasing functions on $\mathbb{R}^n$ with values in $\mathbb{C}^m$. Φ extends by duality to $\mathscr{S}'$ the space of tempered distributions and restricts to the Hilbert space $L_2(\mathbb{R}^n, \mathbb{C}^m)$, the space of Lebesgue measurable square integrable functions, as a unitary map (see [4]) with adjoint-inverse:

$$(\Phi^* f)(p) = (\Phi f)(-p). \tag{1.3}$$

For $p \in \mathbb{R}^n \backslash \{0\}$ and $D_j = -i(\partial/\partial x_j)$ substitute p_j, the jth coordinate of p for D_j, the jth coordinate of D, to obtain the symbol of $A(D)$, $A(p)$. Define ($\mathscr{D}$ = domain)

$$\begin{aligned} \mathscr{D}(A(D)) &= \{ f \in L_2(\mathbb{R}^n, \mathbb{C}^m) \mid A(p)\hat{f}(p) \in L_2 \}. \\ A(D) &= \Phi^* A(p)\, \Phi \qquad \text{on} \quad \mathscr{D}(A(D)). \end{aligned} \tag{1.4}$$

It is well known that $A(D)$ is then a selfadjoint operator on $L_2(\mathbb{R}^n, \mathbb{C}^m)$. We write L_2 for $L_2(\mathbb{R}^n, \mathbb{C}^m)$.

By $\lambda_i(p)$ we mean the solutions to the equation (det = determinant)

$$\det(A(p) - \lambda I_{m \times m}) = 0. \tag{1.5}$$

for fixed p, order these as (they are real by symmetry) (see Appendix C)

$$\lambda_l(p) \geqslant \lambda_{l-1}(p) \geqslant \cdots \geqslant \lambda_1(p) \geqslant \lambda_0(p) = \geqslant \lambda_{-1}(0) \geqslant \cdots \geqslant \lambda_{-l}(p). \tag{1.6}$$

Write $\omega = p/|p|$, $p = |p|\,\omega$, $\omega \in S^{n-1}$. It is known [5] that the ordering (1.6) and (1.7) uniquely specifies the $\lambda_j(p)$ with the possible exception of an

S^{n-1} null set of ω values. By the symmetry of $A(p)$, and since $A(p)$ is first order we may have

$$\begin{aligned} \lambda_j(-p) &= \lambda_{-j}(p) = -\lambda_j(p) \\ \lambda_j(\alpha p) &= \alpha\lambda_j(p), \qquad \alpha > 0. \end{aligned} \tag{1.7}$$

The existence of a $\lambda_j \equiv 0$ on a nontrivial set *is a fact of life for the wave equations of classical physics* and therefore $A(D)$ is not elliptic. *We shall assume that $\lambda_j(p)$ is a C^2 function of $p \neq 0$.*

Further we assume $|p|/\lambda_j(p) < \infty$ ($A(D)$ is strongly propagative). $\tilde{\gamma}_j$ will be a small neighborhood of O in $\mathbb{C}$. Our assumptions imply that the slowness surfaces

$$\{p \mid \operatorname{sig}(j)\, \lambda_j(p) = 1\} = S_j \tag{1.9}$$

are smooth enough so that generalized polar coordinates may be defined on them. Define

$$\hat{P}_j(p) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_j(p)| = \delta_j(p)} (A(p) - \lambda I_{m \times m})^{-1}\, d\lambda,$$

where $\delta_j(p)$ is small enough to include λ_j only.

By symmetry, $\{\hat{P}_j(p)\}$ is a complete set of orthoprojectors for $A(D)$, $\{\lambda_j(p)\}$,

$$\sum_{j=-l}^{l} \hat{P}_j(p) = I_{m \times m}, \; A(p)\, \hat{P}_j(p) = \lambda_j(p)\, \hat{P}(p). \tag{1.10}$$

We write $\bar{P}_i = \hat{P}_i + \hat{P}_{-i}$ and $\mathscr{H} = L_2(\mathbb{R}^n, \mathbb{C}^m)$. Then $\Phi^* \bar{P}_i(p)\, \Phi = P_i$ is a selfadjoint projector on $\mathscr{H}$.

We define the "energy" partition of $\mathscr{H}$ as

$$\mathscr{H} = \oplus P_i \mathscr{H} = \oplus \mathscr{H}_i. \tag{1.11}$$

This is well defined since $\hat{P}_j$ is a measurable function of p, homogeneous of order zero which can also be written ($i \neq 0$) [3]

$$d\hat{P}_i(p) = \sum_{j=1}^{c_i} f_j^i(p) \otimes f_j^i(p), \tag{1.12}$$

where the family $\{f_j^i(p)\}_{j=1}^{c_i}$ is a complete set of (measurable [8]) orthonormal eigenvectors for λ_i (having multiplicity c_i); $\otimes$ is the Kronecker product. d is a normalizing factor.

We define the associated "energy" spaces E_j by the $\|\cdot\|_{E_j}$ norm completion of $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^{c_j})^2$,

$$\|f\|_{E_j} = \int_{\mathbb{R}^n} (\lambda_j(p))^2 \, |\hat{f}_1(p)|^2 + |\hat{f}_2(p)|^2 \, dp; \tag{1.13}$$

E_j is a Hilbert space.

In the same fashion as before, define

$$\Sigma_j = i\Phi^* \begin{bmatrix} 0 & I_{c_j \times c_j} \\ -\lambda_j(p)^2 \, I_{c_j \times c_j} & 0 \end{bmatrix} \Phi \tag{1.14}$$

on E_j.

The main result of [3] is that there exists a unitary map τ

$$\tau = \bigoplus_{j \neq 0} E_k \to \mathscr{H} \ominus \mathscr{H}_0, \tag{1.15}$$

where $\tau(\mathscr{D}(\oplus \Sigma_j)) = \mathscr{D}(A(D))$ and $\tau = E_j \to H_j$ $(j \neq 0)$. Σ_j is simply the anisotropic version of (0.2) (spatial part).

If T is an operator on a normed space, then $\sigma(T)$, $\rho(T)$ denote the spectrum and resolvent sets for T.

$\mathscr{B}(X, Y)$ denotes the space of bounded operators from X to Y with uniform norm. $C(X, Y)$ denotes the compact operators with uniform norm.

Our plan will be to construct a proof of a limiting absorption principle for (0.3), mapping back to (0.1) via τ.

2. Properties of Resolvents

Define the operator $\lambda_j(D)^2$ on $L_2(\mathbb{R}^n, \mathbb{C})$ by $\Phi^* \lambda_j^2(p) \, \Phi$ in the usual way. It is easily seen that $\sigma(\lambda_j(p)^2)$ is the set $[a, \infty]$, where $a = \inf\{\lambda_j(p)^2, p \in \mathbb{R}^n\}$, thus $a = 0$. We define certain other Hilbert spaces and spaces of distributions:

$$L_{2,\alpha} = \left\{ f \,\middle|\, \int_{\mathbb{R}^n} (1 + |x|^2)^\alpha \, |u(x)|^2 \, dx < \infty \right\} \tag{2.1}$$

$$(\mathscr{H} \ominus \mathscr{H}_0)_{\text{loc}} = \{ v \in \mathscr{S}' \mid E_A v \in \mathscr{H} \ominus \mathscr{H}_0, \, P_0 E_A v = 0 \}. \tag{2.2}$$

Here A is some compact set in $\mathbb{R}^n$, ψ_A is a smooth function whose support is in A and $E_A = \Phi^* \psi_A \Phi$. $(\mathscr{H} \ominus \mathscr{H}_0)_{\text{loc}}$ is a Frechet space with the obvious seminorms.

$L_{2,\alpha}$ is the Fourier transform of the Sobolev space $\mathscr{H}_\alpha$. We define the weighted spaces $\mathscr{H}_{\alpha,\beta}$ by the norm

$$\|u\|_{\alpha,\beta}^2 = \int_{\mathbb{R}^n} (1+|x|^2)^\alpha \sum_{|m| \leqslant \beta} |D^m u|^2 \, dx. \tag{2.3}$$

Then $u \in \mathscr{H}_{\alpha,\beta}$ if and only if $\hat{u} \in \mathscr{H}_{\beta,\alpha}$. It appears from (2.3) that α, β must be integers but of course since $u \in \mathscr{H}_\alpha \Leftrightarrow \hat{u} \in L_{2,\alpha}$, $\mathscr{H}_{\alpha,\beta}$ is easily generalized to all real α, β. Note $\mathscr{H}_2 = \mathscr{H}_{0,2}$.

The norm in $\mathscr{H}_\alpha$ will be written $\|\cdot\|_{0,\alpha}$, while the norm in $L_{2,\beta}$ is written $\|\cdot\|_\beta$.

Our first lemma concerns the operator $\lambda_j(D)$.

LEMMA 2.1. *For $\alpha > \frac{1}{2}$, $u \in \mathscr{D}(\lambda_j(D)^2)$, $\lambda \in \mathbb{C} \backslash (\mathbb{R}_+ \cup \tilde{\gamma}_j)$,*

$$\|u\|_{-\alpha,2} \leqslant C \, \|((\lambda_j(D))^2 - \lambda) \, u\|_\alpha. \tag{2.4}$$

The constant C does not depend on λ, for λ in the indicated region.

Proof. The proof is found in Appendix A.

LEMMA 2.2. *For $\lambda \in \mathbb{C} \backslash \tilde{\gamma}_j$ the resolvent $(\lambda_j(D)^2 - \lambda)^{-1}$ has boundary values $(\lambda_j(D)^2 - \lambda)_{\pm}^{-1}$ on $\mathbb{R}_+ \backslash \tilde{\gamma}_j$ in the sense that $\lim_{\pm \operatorname{im} \lambda \to 0, \operatorname{im} \lambda > 0}$ $(\lambda_j(D)^2 - \lambda)^{-1}$ exists as a bounded operator in $\mathscr{B}(L_{2,\alpha}, \mathscr{H}_{2,\alpha})$ $(\alpha > \frac{1}{2})$ and the maps $\lambda \to (\lambda_j(D)^2 - \lambda)^{-1}$, $\lambda \to (\lambda_j(D)^2 - \lambda)_{\pm}^{-1}$ are continuous $B(L_{2,\alpha}, \mathscr{H}_{\alpha,2})$ valued maps and continuous $C(L_{2,\alpha}, L_2, -\alpha)$ valued maps.*

Proof. The proof is found in Appendix B. (In fact, these are Holder continuous maps.)

3. PERTURBATIONS

Consider the operators Σ_j.

LEMMA 3.1. *Suppose $\lambda \notin \mathbb{R}$. Then*

$$(\Sigma_j - \lambda)^{-1}$$

$$= \Phi^* \begin{bmatrix} \dfrac{C_1(\lambda)}{\lambda_j^2 - \lambda^2} & \dfrac{C_2}{\lambda_j^2 - \lambda^2} I_{c_j \times c_j} \\ \dfrac{\lambda_j^2 C_3(\lambda)}{\lambda_j^2 - \lambda^2} I_{c_j \times c_j} & \dfrac{C_1(\lambda)}{\lambda_j^2 - \lambda^2} \end{bmatrix}_{2c_j \times 2c_j} \Phi$$

$$= \begin{bmatrix} C_1(\lambda) & C_2 I_{c_j \times c_j} \\ C_3(\lambda) I_{c_j \times c_j} & C_1(\lambda) \end{bmatrix}_{2c_j \times 2c_j} (\lambda_j(D)^2 - \lambda^2)^{-1} + \begin{bmatrix} O_{c_j \times c_j} & O_{c_j \times c_j} \\ I_{c_j \times c_j} & O_{c_j \times c_j} \end{bmatrix}, \tag{3.1}$$

where $C_i(\lambda)$ is bounded on compact subsets of $\mathbb{C}$.

Proof. Induction on the size of c_j. The details are left to the reader.

COROLLARY 3.2. *The spectrum of Σ_j is $\mathbb{R}$.*

It is easy to see that $(\Sigma_j - \lambda)^{-1}: E \to \mathscr{D}(\Sigma_j)$. Note that since $n > 2$, E_j consists of functions defined almost everywhere.

LEMMA 3.3. *Suppose $C(x)$ is a bounded measurable matrix-valued function. Then $\Phi^* C(X)\,\Phi$ is a bounded operator on $L_{2,\alpha}$ for all real α.*

Proof. This is an elementary consequence of the fact that $L_{2,\alpha}$, $L_{2,-\alpha}$ are dual via the L_2 scalar product and that the functions $P(k)\,e^{-k^2/2}$ are dense $P(k)$ a polynomial in k) in the spaces $L_{2,\alpha}$, $L_{2,-\alpha}$ and are invariant under Fourier transform.

Now we are able to study the general anisotropic form of (0.3). Let us assume that K_j is a matrix-valued function of size $2c_j \times 2c_j$. Unlike Schrodinger operators, a natural division between short and long range asympotics lies at exponent 2 instead of 1. That is

$$K_j = O(|x|^{-\gamma}), \qquad |x| \to \infty$$

is *short range* if $\gamma > 2$ and *long range* otherwise. This is simply a result of energy considerations for $\Sigma_j + K_j$. However, since $\Sigma_j + K_j$ for most physically interesting cases is *not* selfadjoint, energy considerations play a small role in the general case. Henceforth, we shall assume

(1) K_j is bounded.

(2) $K_j \sim O(|x|^{-1-\varepsilon})$, $\varepsilon > 0$ (long range)?

(3) The first c_j columns of K_j have L_∞ first derivatives.

(4) Let (k^j_{lm}) be the first c_j rows of K_j. We assume the determinant

$$\det \begin{bmatrix} 1 & & & & & & \\ & 1 & & 0 & & & \\ & & 1 & & & & \\ 0 & & & 1 & & & \\ & & & & \ddots & & \\ k_{11} & \cdots & \cdots & \cdots & k_{lm}+1 & \cdots & k_{1c_j} \\ \vdots & & & & & & \\ k_{c_j 1} & \cdots & \cdots & \cdots & \cdots & \cdots & k_{c_j c_j}+1 \end{bmatrix} \tag{3.2}$$

is locally nonzero in $\mathbb{R}^n$.

Remarks. Weaker assumptions are possible. (1) can be weakened obviously. (2) may possibly be weakened by considering the Hermitian and anti-Hermitian parts of K_j. By localizing, (4) may be weakened but weaker results are obtained (see [7]).

The limiting absorption principle for $\Sigma_j + K_j$ will now be established. This in turn will (1) identify corresponding "natural" perturbations of $A(D)$ in the setting of (0.1) and (2) establish the existence of steady-state solutions for them.

We are interested in solutions of the equation

$$i\begin{pmatrix} 0 & I_{c_j \times c_j} \\ -\lambda_j(D)^2 I_{c_j \times c_j} & 0 \end{pmatrix}_{2c_j \times 2c_j} u + K_j u - \lambda n = f \tag{3.3}$$

for $\operatorname{im} \lambda = 0$. Thus equivalently,

$$((\Sigma_j - \lambda)^{-1} K_j + I)\, u = (\Sigma_j - \lambda)^{-1} f \tag{3.4}$$

or

$$u(x, \lambda) = ((I + (\Sigma_j - \lambda)^{-1} K_j)^{-1} (\Sigma_j - \lambda) f(x). \tag{3.5}$$

LEMMA 3.3. *$(I + (\Sigma_j - \lambda)^{-1} K_j)^{-1}$ exists as a bounded operator on $L_{2, -\alpha}$ with $\frac{1}{2} < \alpha < \frac{1}{2} + \delta$ for δ small enough, and $\lambda \notin M_\pm \cup \tilde{\gamma}_i$, where $M_\pm$ is discrete in $\mathbb{C} \backslash \mathbb{R}$ and has linear measure zero in $\mathbb{R}$.*

Remark. To be precise, we actually mean that $(I + (\Sigma_j - \lambda)^{-1} K_j)^{-1}$ has two boundary values in $\mathbb{R} \backslash (M_\pm \cup \tilde{\gamma}_j)$ on the upper and lower "edges" of $\mathbb{R} \backslash (M_\pm \cup \tilde{\gamma}_j)$. It is single-valued in $\mathbb{C}_+ \backslash M_+$ and in $\mathbb{C}_- \backslash M_-$. Without loss of generality, we assume $\operatorname{im} \lambda > 0$.

Proof of Lemma 3.3.

$$\begin{aligned}
&(\Sigma_j - \lambda)\, k_j \\
&\quad = \left(\begin{bmatrix} c_1(\lambda) & c_2 I \\ c_3(\lambda) I & c_1(\lambda) \end{bmatrix} (\lambda_j(D)^2 - \lambda^2)^{-1} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \right) K_j && (3.6) \\
&\quad = [c_{lm}(\lambda)](\lambda_j(D)^2 - \lambda^2)^{-1} K_j + \tilde{K}_j. && (3.7)
\end{aligned}$$

Notice $K_j = L_{2, -\alpha} \to L_{2, \alpha}$ by assumption (2) for α close to $\frac{1}{2}$. Thus $(\lambda_j(D)^2 - \lambda^2)^{-1} K_j$ is a compact (Lemma 2.2) operator with compact boundary values on $\mathbb{R} \backslash \tilde{\gamma}_j$. Further, by assumptions (1), (2), and (4), the operator $(I - \tilde{K}_j)^{-1}$ is bounded on all the spaces $L_{2, -\beta}$. Note the resolvent identity

$$\begin{aligned}
&(I + (\Sigma_j - \lambda)^{-1} K_j)^{-1} \\
&\quad = (I + \tilde{K}_j) + (C_{lm}(\lambda))(\lambda_j(D)^2 - \lambda^2)^{-1} K_j)^{-1} \\
&\quad = (I + \tilde{K}_j)^{-1} + (I + \tilde{K}_j)^{-1} (I + (I + \tilde{K}_j)^{-1} C_{lm}(\lambda)(\lambda_j(D)^2 - \lambda^2)^{-1} K_j)^{-1}.
\end{aligned} \tag{3.8}$$

Applying the Fredholm theory to the term

$$(I+(I+\tilde{K}_j)^{-1}C_{lm}(\lambda)(\lambda_j(D)^2-\lambda^2)^{-1}K_j)^{-1} \tag{3.9}$$

shows (see [7]) that inverse (3.9) exists except possibly for a discrete set in $\mathbb{C}_+ = \{\lambda \mid \operatorname{im}\lambda > 0\}$ and a nowhere-dense set of measure zero in $\mathbb{R}\backslash\tilde{\gamma}_j$. (We may shrink $\tilde{\gamma}_j$ as much as we like.) The proof for im $\lambda \leqslant 0$ is identical. This completes the proof.

THEOREM 3.4. *Suppose* $f \in L_{2,\alpha}$, $\lambda \in \mathbb{R}\backslash\{M^j_\pm \cup \tilde{\gamma}_j\}$ *and* K_j *satisfies* (1)–(4). *Then there exists* $u_\pm \in L_{2,-\alpha}$ *so that*

$$\Sigma_j u_\pm + K_j u_\pm - (\lambda \pm iO)\, u_\pm = f. \tag{3.10}$$

Proof. By Lemma 3.3 and Eq. (3.5) we obtain (3.10).

Now we note that (1)–(4) imply that $\Sigma_j + K_j : \mathscr{D}(\Sigma_j) \to E_j$. Therefore

$$(\Sigma_j + K_j - \lambda)^{-1} : E_j \to \mathscr{D}(\Sigma_j)$$

thus

$$(\Sigma_j + K_j - \lambda)^{-1} : E_j \cap L_{2,\alpha} \to \mathscr{D}(\Sigma_j) \cap L_{2,-\alpha}$$

and therefore $E_A \tau u_\pm$ are well defined for $f \in E_j \cap L_{2,\alpha}$, since the symbol of τ, $\hat{\tau}$ is $\sim \lambda_j(p)$ (see [3]). Clearly $E_A \tau u_\pm \in (\mathscr{H} \ominus \mathscr{H}_0)_{\text{loc}}$ and $\tau u_\pm$ is a solution (in distribution sense) of

$$\tau \Sigma_j \tau^* v + \tau K_j \tau^* v - \lambda v = g, \tag{3.11}$$

$g = \tau f$. Therefore we have

THEOREM 3.5. *Let* $g = \tau f$, *where* $f \in E_j \cap L_{2,\alpha}$. *Suppose* $\lambda \in \mathbb{R}\backslash(\bigcup_{j=1}^{l} M^l_\pm(\bigcup_j \tilde{\gamma}_j))$. *Then there exist incoming and outgoing solutions* $v_\pm$ *to*

$$A(D)\, v_\pm + Bv_\pm - \lambda \leftrightarrow (\lambda \pm iO)\, v_\pm = g, \tag{3.12}$$

where $v_\pm \in (\mathscr{H} \ominus \mathscr{H}_0)_{\text{loc}}$, $B = \tau \oplus K_j \tau^*$.

4. EXTENSIONS

The results of the preceding sections indicate that perhaps one underlying reason for the study of Eq. (0.3) (in practical terms) is that the full dispersion is too unwieldy. However, the proper form of the dispersion is revealed by Eq. (3.12).

It is easy to imagine though that a model like (3.12) does not completely describe a given physical situation. Surely (though it may not be relevant in a given problem) there will be transfer of energy between the $\mathscr{H}_j$, even if given assumptions on the problem make it very small so that it may be "neglected." Just as a boundary in an elastic medium may cause incident pressure and shear waves to become coupled, a dispersion may cause the unperturbed incoming and outgoing waves to become coupled by "reflection," etc.

Let us then briefly consider the parrtial "inverse" problem: If the dispersion couples (in a sufficiently weak manner) the energy spaces $\mathscr{H}_i$, does there exist a steady-state solution of (0.1) (see again [6] for pointwise dispersion)?

We shall suppose that energy transfer is small in the sense that if G_{ij} is a transfer from $\mathscr{H}_i \to \mathscr{H}_j$ then G_{ij} has compact support and bounded operator norm. The operators $\tau G_{ij} \tau^* = E_i \to E_j$ then have the properties

$$\begin{aligned} &(1) \quad \tau G_{ij} \tau^* = L_{2,-\alpha} \to L_{2,\alpha} \\ &(2) \quad \| \oplus \tau G_{ij} \tau^* f \|_{L_{2,-\alpha}} < C_{ij}. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *Suppose the conditions* (4.1) *hold. Then steady-state solutions of* (0.1) *exist in* $(\mathscr{H} \ominus \mathscr{H}_0)_{\mathrm{loc}}$.

Proof. Suppose $f \in \oplus L_{2,\alpha}$, $\alpha > \frac{1}{2}$, etc. and $i, j \neq 0$:

$$(\oplus \Sigma_j + \oplus K_j + \oplus \tau G_{ij} \tau^* - \lambda)\, u = f \tag{4.2}$$

$$[I + (\oplus \Sigma_j - \lambda)^{-1} (\oplus K_j + \oplus \tau G_{ij} \tau^*)]\, u = (\oplus \Sigma_j - \lambda)^{-1} f. \tag{4.3}$$

The operator [ ] in (4.3) has an inverse in an $L_{2,-\beta}$ space for β close to $\frac{1}{2}$. Thus τu is a solution in $(\mathscr{H} \ominus \mathscr{H}_0)_{\mathrm{loc}}$.

The terms of the form G_{ij} for i or $j = 0$ are excluded. They cause a difficulty for low frequencies and the general case of (0.1) fails to have a steady-state solution for low frequencies (depending on B), see [6].

SUMMARY

The purpose of this work has been to extend the classical models (0.1) to certain problems arising in modern wave propagation theory and to prove the existence of steady-state solutions for these equations.

These questions have been studied by means of perturbation theory of certain pseudodifferential operators.

It would be desirable to extend these results to domains exterior to some bounded set or to a halfspace. It would also be of interest to study more general conditions on K_j.

APPENDIX A: THE PROOF OF LEMMA 2.1

LEMMA 2.1. *For* $\alpha > \frac{1}{2}$, $u \in \mathscr{D}(\lambda_j(D)^2)$, *and* $\lambda \in \mathbb{C} \backslash (\mathbb{R}_+ \cup \tilde{\gamma}_j)$,

$$\|u\|_{-\alpha,2} \leqslant C\, \|((\lambda_j(D))^2 - \lambda)\, u\|_\alpha .$$

The constant C does not depend on λ *for* λ *in the indicated region.*

It is possible to give a direct proof, but we avoid this and take a short cut.

Proof. First we note that an easy consequence of Weder [9, Eq. (2.9)] is that

$$\|(A(D) - \lambda)\, u\|_\alpha \geqslant C\, \|u\|_{-\alpha,1},\quad u = (I - P_0)\, u,$$

where $u \in \mathscr{D}(A(D))$. Note also that since

$$A(p)\, P_j(p) = \lambda_j(p)\, P_j(p),$$

we have (assume without loss that $j \neq 0$)

$$\|(A(D) - \lambda)\, P_j u\|_\alpha \geqslant C\, \|P_j u\|_{-\alpha,1}$$

or supposing $u_j \in \mathscr{H}_j$,

$$\|(\lambda_j(D) - \lambda)\, u_j\|_\alpha \geqslant C\, \|u_j\|_{-\alpha,1}.$$

Since for this argument $A(D)$ is arbitrary (only λ_j is fixed) we may take $\lambda_j^2 = \lambda_k^2$ for all $k \neq 0$. Thus

$$\|(\lambda_j(D) - \lambda)\, u\|_{+\alpha} \geqslant C_1\, \|u\|_{-\alpha}$$

for all u. This shows that since

$$\begin{gathered} \lambda_j(D)^2 - \lambda^2 = (\lambda_j(D) - \lambda)(\lambda_j(D) + \lambda), \\ \|(\lambda_j(D)^2 - \lambda^2)\, u\|_\alpha \geqslant \bar{C}_1\, \|u\|_{-\alpha,2}, \end{gathered} \tag{A.1}$$

where $\bar{C}_1$ depends on $\tilde{\gamma}_j$ but not on λ. This is sufficient for the lemma.

APPENDIX B: THE PROOF OF LEMMA 2.2

LEMMA 2.2. *For* $\lambda \in \mathbb{C} \backslash \tilde{\gamma}_j$ *the resolvent* $(\lambda_j(D)^2 - \lambda)^{-1}$ *has boundary values* $(\lambda_j(D)^2 - \lambda)_{\pm}^{-1}$ *on* $\mathbb{R}_+ \backslash \tilde{\gamma}_j$ *in the sense that* $\lim_{\pm \operatorname{im} \lambda \to 0,\, \operatorname{im} \lambda > 0}$ $(\lambda_j(D)^2 - \lambda)^{-1}$ *exists as a bounded operator in* $\mathscr{B}(L_{2,\alpha}, \mathscr{H}_{2,\alpha})$ $(\alpha > \frac{1}{2})$ *and* *the maps* $\lambda \to (\lambda_j(D)^2-\lambda)^{-1}$ *and* $\lambda \to (\lambda_j(D)^2-\lambda)_{\pm}^{-1}$ *are continuous* $B(L_{2,\alpha}, \mathscr{H}_{\alpha,2})$*-valued maps and continuous* $C(L_{2,\alpha}, L_{2,-\alpha})$*-valued maps.*

Proof. Let $f, g \in L_{2,\alpha}$. Then for $\lambda \in K \Subset \backslash\{\mathbb{R} \cup \tilde{\gamma}_j\}$:

$$\|(\lambda_j(D)^2-\lambda)^{-1} f\|_{-\alpha} \leqslant C \|f\|_\alpha \tag{B.1}$$

and

$$|((\lambda_j(D)^2-\lambda)^{-1} f, g)| \leqslant C \|f\|_\alpha \|g\|_\alpha \tag{B.2}$$

by Lemma 2.1.

Now suppose $f, g \in C_0^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
&((\lambda_j(D)^2-\lambda)^{-1} f, g) \\
&= \int_{\mathbb{R}^n} \frac{\hat{f}(p)\,\hat{g}(p)}{\lambda_j(p)^2-\lambda}\, dp \\
&= \frac{1}{2} \int_0^\infty \frac{t^{(n-2)/2}}{t-\lambda} \int_{|\lambda_j(p)|=1} \hat{f}(t^{1/2}\lambda_j(p))\,\hat{g}(t^{1/2}\lambda_j(p))\, d\sigma_j(p)\, dt.
\end{aligned}$$

Where we have used the fact that $p \to \lambda_j(p)$ are C^1 etc., σ_j being the appropriate surface measure, $d\sigma_j = dS_j/\sqrt{|\nabla\lambda_j(p)|}$. dS_j being the surface element of the jth *slowness surface* [5]. It follows from Cauchy principle value theory, that (in K)

$$\lim_{\pm \operatorname{im} \lambda \to 0^+} ((\lambda_j(D)^2-\lambda)^{-1} f, g) \qquad \text{exists}$$

and is continuous in $\lambda \in \bar{K}$ having, perhaps, different values on the upper and lower edges of $\mathbb{R}_+ \backslash \tilde{\gamma}_j$. By duality, we therefore have that

$$\lim_{\pm \operatorname{im} \lambda \to 0} (\lambda_j(D)^2-\lambda)^{-1} f \qquad \text{exists} \tag{B.3}$$

weakly in $L_{2,-\alpha}$. By Lemma 2.1, $\lambda \in K$ implies that

$$(\lambda_j(D)^2-\lambda)^{-1} f \in \mathscr{H}_{-\alpha,2}.$$

Since $\mathscr{H}_{-\alpha,2}$ is a Hilbert space, bounded sets are weakly compact. Thus the weak limit (B.3) is in $\mathscr{H}_{-\alpha,2}$. Thus, by the Riesz theorem (B.3) defines a bounded operator(s) $(\lambda_j(D)^2-\lambda)_\pm : L_{2,\alpha} \to \mathscr{H}_{-\alpha,2}$. Now let $\lambda_j \in K$, $\lambda_j \to \lambda \in \bar{K} \cap \bar{\mathbb{C}}_+$ (without loss of generality, assume $\lambda \in \mathbb{R}_+$). Then since $\mathscr{H}_{-\alpha,2} \subsetneq L_{-\alpha,2}$ is a compact mapping (Rellich)

$$\lim_{\lambda_j \to \lambda} (\lambda_j(D)^2-\lambda)^{-1} f \qquad \text{exists}$$

in the $L_{-\alpha,2}$ sense. Taking $f_i \to f$ in the weak sense we can show from (B.2) that

$$\lim_{i \to \infty} (\lambda_j(D)^2 - \lambda_i)^{-1} f_i \qquad \text{exists}$$

in the weak sense. Repeating the steps above shows that the limit exists in the strong sense. An easy reductio ad absurdum shows in fact that $(\lambda_j(D)^2 - \lambda_i)^{-1} \to (\lambda_j(D) - \lambda)_+^{-1}$ in the uniform topology, $L_{2,+\alpha} \to L_{2,-\alpha}$. Now it can be checked that

$$(\lambda_j(D)^2 - \lambda)_{\pm}^{-1} = L_{2,\alpha} \to \mathscr{H}_{-\alpha,2},$$

is continuous in λ for we must simply show that ($\Delta =$ Laplacian)

$$\Delta(\lambda_j(D)^2 - \lambda)_+^{-1} = L_{2,\alpha} \to L_{2,-\alpha} \tag{B.4}$$

is continuous in λ. But $\lambda_j^2(D)/\Delta$ is bounded and invertible on $L_{2,-\alpha}$ (recall $A(D)$ is strongly propagative) so it suffices to check that $\lambda_j(D)^2(\lambda_j(D)^2 - \lambda)_+^{-1}$ has the property (B.4). This is so because $\lambda_j(D)^2(\lambda_j(D)^2 - \lambda)^{-1} = I + \lambda(\lambda_j(D)^2) - \lambda)^{-1}$. Application of the Rellich compactness theorem completes the result for the last statement of Lemma 2.2.

Appendix C

We noted the inequalities

$$\lambda_l \geqslant \lambda_{l-1} \geqslant \cdots \geqslant 0 \geqslant \lambda_{-1} \geqslant \cdots \geqslant \lambda_{-l}$$

in (1.6). The following simple proof of this was pointed out to the author by Professor Wayne Barrett.

Let S be a real $n \times n$ skew-symmetric matrix and suppose

$$A = \begin{bmatrix} 0 & S \\ S^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & S \\ -S & 0 \end{bmatrix}_{2n \times 2n}$$

Since A is symmetric, its eigenvalues are real. Suppose n is odd and write $n = 2k + 1$ with the eigenvalues of S being

$$\pm ip_1, \ \pm ip_2, \ \pm ip_3, \ldots, \ \pm ip_k, \ 0.$$

But

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes S$$

where

$$\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \{i, -i\}.$$

Using the facts

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

and thus for $Ax = \lambda_1 x$ and $By = \lambda_2 y$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda_1 x \otimes \lambda_2 y = \lambda_1 l_2(x \otimes y).$$

We have that the eigenvalues of A are

$$\pm p_1, \pm p_2, \pm p_3, ..., \pm p_k, 0.$$

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